A New Approach to the Study of Coupled Nonlinear Systems

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In this paper, a new approach to the problem of nonlinear coupling has been presented. This approach is based on differential transformation techniques which have been successfully used elsewhere in the literature for the study of nonlinear systems in uncoupled form. Here, the object of these transformations is to reduce the given set of coupled ordinary differential equations to an equivalent set of uncoupled equations amenable to existing techniques. The method is illustrated with an example drawn from the field of space mechanics.

1. Introduction

IT is well known that the governing differential equations in the field of linear mechanics can always be obtained in an uncoupled form, since the kinetic and potential energy quadratics can be simultaneously reduced to their normal form and the application of the Lagrange's equation leads to a system of uncoupled equations. A parallel method for systems in the nonlinear domain is not known. Also given a pair of coupled linear differential equations, suitable algebraic transformations of the variables, which result in uncoupling of the equations, can be thought of. But in general this is not a feasible proposition for nonlinear problems. One possible approach to the study of such coupled nonlinear systems, is to obtain a higher-order uncoupled system by differentiation and mutual substitution of the given set of coupled differential equations. But the main disadvantage of this approach is the difficulties, encountered in studying higher order, nonlinear systems, which are too well known. Hence, attempts were made to develop suitable transformation techniques for the study of coupled nonlinear systems and this paper sets out the results of the research pursued in this direction.

2. Analysis

A coupled dynamical system in two variables, say x_1 and x_2, can, in general, be thought of as spanning a three-dimensional space (x_1,x_2,t). In the case of linear systems, the uncoupling of this coupled system is effected by obtaining the projection of this system on the two constituent planes (x_1,t) and (x_2,t) of the space (x_1,x_2,t) considered earlier. This direct approach is not practicable in the case of nonlinear systems with or without variable parameters. This calls for a suitable alternate approach to the problem. One such approach is based on the idea of effecting the uncoupling through a study of the system in the third constituent plane, (x_1,x_2) [of the three dimensional space (x_1,x_2,t)] or its transformations. Once the relation between x_1 and x_2 is established, the resulting uncoupled systems can be studied through known techniques.

This relatively unconventional approach, though not universal in application, can be extended to cover a fairly wide range of problems, leading to coupled, nonlinear, nonautonomous systems, depending on the ingenuity of the user.

2.1 Second-Order Systems

At the first instance, consider, a coupled nonlinear, nonautonomous system of second order, which can be represented by a pair of simultaneous first-order differential equations with variable coefficients of the type,

$$f_1(x_1,x_2,\dot{x}_1,t) = 0 \tag{1a}$$

$$f_2(x_2,x_1,\dot{x}_2,t) = 0 \tag{1b}$$

Here x_1 and x_2 are the two dependent variables which represent two properties of the behavior of the system, such as displacement and velocity of a dynamical system. The dots represent the differentiation with respect to time t the independent variable. Now, the preceding pair of simultaneous equations represent the given system in a three dimensional space (x_1,x_2,t). As stated earlier, the projection of the system on the two planes (x_1,t) and (x_2,t) represent the two uncoupled equations in x_1 and x_2 separately. In the case of nonlinear systems, it is difficult to obtain this uncoupling directly and a new approach is made.

By the application of the method, which primarily concentrates on the idea of studying the system in the (x_1,x_2) plane [the third constituent plane of the space (x_1,x_2,t)], a differential equation in (x_1,x_2) is obtained, with t being eliminated both explicitly and implicitly. Either of the variables x_1 and x_2 can be chosen as the new independent variable depending on the mathematical convenience of the particular problem on hand. For purposes of analysis, let one of the variables say x_2 be the independent variable.

This leads to the differential transformation law

$$\dot{x}_1 = dx_1/dt = (dx_1/dx_2)(dx_2/dt) = x'_1\dot{x}_2 \tag{2}$$

The prime indicates differentiation with respect to x_2. Substituting Eq. (2) in Eq. (1a) gives

$$g(x_1,x_2,x'_1,\dot{x}_2,t) = 0 \tag{3}$$

It can be observed that Eq. (3) will be an uncoupled differential equation in x_1 with variable coefficients (in terms of x_2) provided t and $\dot{x}_2$ are eliminated from it. An obvious way of doing it would be to eliminate simultaneously these variables between Eqs. (1b) and (3). It is to be noted that this elimination process is not always possible since there must be at least three independent equations to eliminate two variables to make the elimination of the variables a certainty. Here, as the number of available equations is only two, a single variable can be eliminated. Hence, if t and $\dot{x}_2$ appear in such a form as to allow it to be treated as a single variable, then the uncoupled equation can be obtained with certainty. This, therefore imposes a condition on the form of the equations for this approach to be successful in case of nonautonomous systems. But in case of autonomous systems, the uncoupling is guaranteed because only $\dot{x}_2$ need be eliminated between the two equations and this can always be achieved.

Assuming such an elimination has been made, Eqs. (3) in the (x_1,x_2) plane, can be written as

$$h(x_1,x_2,x'_1) = 0 \tag{4}$$

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which is an uncoupled nonlinear/linear differential equation with variable coefficients. This on being solved by other existing techniques, gives the solution

$$x_1 = p(x_2) \tag{5}$$

Substituting Eq. (5) in Eqs. (1a) and (1b) gives the two uncoupled equations,

$$q_1(x_1,\dot{x}_1,t) = 0 \tag{6a}$$

$$q_2(x_2,\dot{x}_2,t) = 0 \tag{6b}$$

which can individually be solved to obtain the two solutions $x_1(t)$ and $x_2(t)$. Having outlined the method, it is necessary to obtain the condition on the form of the equations, for the elimination of t and $\dot{x}_2$ to be feasible in the case of nonautonomous systems.

Writing the Eqs. (1a) and (1b) in the form

$$\dot{x}_1 + g_1(x_1,x_2,t) = 0 \tag{7a}$$

$$\dot{x}_2 + g_2(x_2,x_1,t) = 0 \tag{7b}$$

and using the Eq. (2), modifies the Eq. (7a) as

$$x'_1\dot{x}_2 + g_1(x_1,x_2,t) = 0 \tag{8}$$

Substituting for $\dot{x}_2$ from Eq. (7b), gives

$$x'_1 - g_1(x_1,x_2,t)/g_2(x_1,x_2,t) = 0 \tag{9}$$

Now Eq. (9) will be an uncoupled equation in x_1 and its derivatives if

$$\begin{aligned} g_1(x_1,x_2,t)/g_2(x_1,x_2,t) &= \text{function of } x_1 \text{ and } x_2 \text{ only} \\ &= a(x_1,x_2) \end{aligned} \tag{10}$$

Hence, the equations of the given system, (for uncoupling to be practicable through this approach) must be of the form

$$\dot{x}_1 + a(x_1,x_2)g(x_1,x_2,t) = 0 \tag{11a}$$

and

$$\dot{x}_2 + g(x_1,x_2,t) = 0 \tag{11b}$$

or rewriting in another form,

$$\dot{x}_1 + a_1(x_1,x_2)b(t) = 0 \tag{12a}$$

$$\dot{x}_2 + a_2(x_1,x_2)b(t) = 0 \tag{12b}$$

when the system is autonomous, the equations reduce to a form which merely represents a general, coupled nonlinear, autonomous system without any restrictions whatsoever on its form.

Eqs. (11a, 11b and 12a, 12b) represent a fairly wide class of second-order, nonlinear, nonautonomous, coupled systems which can be uncoupled by this technique. [The first pair of equations represent a more general class of system than the second pair which is just a particular case of the first with $g(x_1,x_2,t) = a_2(x_1,x_2)b(t)$.]

The uncoupled equation resulting from the above pair of equations on application of this technique, will be of the form

$$x'_1 = a_1(x_1,x_2)/a_2(x_1,x_2) = a(x_1,x_2) \tag{13}$$

This equation is a first-order, uncoupled, nonlinear differential equation, with variable coefficients and may be solved by existing techniques.[1-3]

As an example, consider a coupled, nonlinear, second-order system, whose equations of motion can be written in the form,

$$\dot{x}_1 + g(t)x_1x_2 = 0 \tag{14a}$$

and

$$\dot{x}_2 + g(t)x_2^2 + g(t)x_1^2x_2 = 0 \tag{14b}$$

For purposes of analysis, make x_1 as the new independent variable

$$\dot{x}_2 = x'_2\dot{x}_1 \tag{15}$$

where prime indicates differentiation with respect to x_1. Substituting Eq. (15) in Eq. (14b) and using Eq. (14a) to eliminate $\dot{x}_1$ in the resulting equation, gives

$$x'_2 = x_2/x_1 + x_1 \tag{16}$$

Eq. (16) is a linear differential equation with variable coefficients and is easily solved to obtain the relation between the variables x_1 and x_2 as

$$x_2 = x_1(C_1 + x_1) \tag{17}$$

Substituting Eq. (17) in (14a) gives the uncoupled equation in x_1 as

$$\dot{x}_1 + g(t)x_1^2(C_1 + x_1) = 0 \tag{18}$$

Equation (18) can be solved by separation of variables to obtain the solution $x_1(t)$ as

$$\left(\frac{x_1}{C_1 + x_1}\right)^{1/C_1^2} \exp\left(\frac{1}{C_1x_1}\right) = C_2 \exp[\textstyle\int g(t)dt] \tag{19}$$

Eq. (19) along with Eq. (17) gives the solution $x_2(t)$, in an implicit form, as

$$\left[\frac{-C_1/2 \pm (C_1^2/4 + x_2)^{1/2}}{C_1/2 \pm (C_1^2/4 + x_2)^{1/2}}\right]^{1/C_1^2} \cdot \exp[1/C_1\{-C_1/2 \pm (C_1^2/4 + x_2)^{1/2}\}] = C_2 \exp[\textstyle\int g(t)dt] \tag{20}$$

The method thus enables the determination of the solutions (in at least implicit forms) to a problem which otherwise is difficult to tackle. Also, the necessary arbitrary constants to fit in the initial conditions are included in the solutions.

2.2 Third-Order Systems

The technique is now extended to other higher-order systems. Consider a third-order system given by the set of equations,

$$\dot{x}_1 + f_1(x_1,x_2,x_3,t) = 0 \tag{21a}$$

$$\dot{x}_2 + f_2(x_1,x_2,x_3,t) = 0 \tag{21b}$$

$$\dot{x}_3 + f_3(x_1,x_2,x_3,t) = 0 \tag{21c}$$

Let x_3 be the new independent variable

$$\dot{x}_1 = x'_1\dot{x}_3 \tag{22a}$$

$$\dot{x}_2 = x'_2\dot{x}_3 \tag{22b}$$

Substitution of Eqs. (22a and 22b) in Eqs. (21a and 21b) and use of Eq. (21c) leads to

$$-f_3(x_1,x_2,x_3,t)x'_1 + f_1(x_1,x_2,x_3,t) = 0 \tag{23a}$$

$$-f_3(x_1,x_2,x_3,t)x'_2 + f_2(x_1,x_2,x_3,t) = 0 \tag{23b}$$

If the given system were to be an autonomous system, then Eqs. (23a) and (23b) would be similar to the equations considered in the previous case viz., a coupled second-order, nonlinear, nonautonomous system, with x_1, x_2 as the dependent variables and x_3 (instead of t) as the independent variable. Hence, the form of the equations must be such as to make t vanish in the above equations, for the method to lead to equations similar to the previous case.

Alternatively, the conditions for the aforementioned equations to be uncoupled can be directly obtained as follows.

For Eq. (23a) to be an uncoupled equation in x_1, the condition is

$$\begin{aligned} f_1(x_1,x_2,x_3,t)/f_3(x_1,x_2,x_3,t) &= \text{function of } x_1 \text{ and } x_3 \text{ only} \\ &= a(x_1,x_3) \end{aligned} \tag{24a}$$

Similarly for Eq. (23b) to be an uncoupled equation in x_2, the condition is

$$\begin{aligned}\frac{f_2(x_1,x_2,x_3,t)}{f_3(x_1,x_2,x_3,t)} &= \text{function of } x_2 \text{ and } x_3 \text{ only} \\ &= b(x_2,x_3)\end{aligned} \tag{24b}$$

Hence, for this method to be applicable, the third-order, nonlinear, nonautonomous system should be of the form,

$$\dot{x}_1 + a(x_1,x_3)c(x_1,x_2,x_3,t) = 0 \tag{25a}$$

$$\dot{x}_2 + b(x_2,x_3)\cdot c(x_1,x_2,x_3,t) = 0 \tag{25b}$$

$$\dot{x}_3 + c(x_1,x_2,x_3,t) = 0 \tag{25c}$$

Depending on the general functions a, b and c, various third-order systems can be represented by the aforementioned equations and thus the method will be applicable for a wide class of problems. For any given problem, the choice of the independent variable is to be judiciously made as, while the choice of one variable may lead to uncoupled equations, the choice of another may not. In this manner, conditions on the form of equations for other higher order systems can be obtained.

Another approach would be to consider the equations for the third-order system in the form of two equations, one of second order and one of first order and this leads to different conditions which describe another class of third order systems that can be tackled by this technique. This approach is now extended to study systems with two degrees of freedom.

2.3 Fourth-Order or Two-Degrees-of-Freedom Systems

A general, coupled, nonlinear, time dependent system with two degrees of freedom can be represented by a pair of simultaneous nonlinear second order differential equations with variable coefficients of the type

$$f_1(x_1,x_2,\dot{x}_1,\dot{x}_2,\ddot{x}_1,t) = 0 \tag{26a}$$

$$f_2(x_2,x_1,\dot{x}_2,\dot{x}_1,\ddot{x}_2,t) = 0 \tag{26b}$$

Here x_1 and x_2, the two dependent variables represent the response of the two-degrees-of-freedom system and the dots indicate differentiation with respect to time t the common independent variable.

As before, the equations represent the given system in a three dimensional space (x_1,x_2,t). The method attempts to achieve the uncoupling by studying the projection of the system on the plane (x_1,x_2) which describes the relation between the variables x_1 and x_2. For purposes of analysis, one of the variable say x_2 is assumed to be the new independent variable. Hence, the differential transformation law becomes,

$$\dot{x}_1 = x'_1\dot{x}_2 \tag{27a}$$

$$\ddot{x}_1 = x''_1\dot{x}_2^2 + x'_1\ddot{x}_2 \tag{27b}$$

Here, the primes indicate differentiation with respect to x_2. Substituting Eqs. (27a) and (27b) in Eqs. (26a) and (26b) gives the equations

$$g_1(x_1,x_2,x'_1,\dot{x}_2,x_1'',\ddot{x}_2,t) = 0 \tag{28a}$$

$$g_2(x_2,x_1,\dot{x}_2,x'_1,\ddot{x}_2,t) = 0 \tag{28b}$$

The Eq. (28a) will be an uncoupled equation in x_1 with variable coefficients (in terms of x_2) provided the variables t, $\dot{x}_2$ and $\ddot{x}_2$ can be eliminated using the Eq. (28b). Obviously, this is possible only if t, $\dot{x}_2$ and $\ddot{x}_2$ appear in such a form as to be treated as a single unit or variable. This imposes certain restrictions as in the previous case on the form of the equations. Assuming such an elimination is possible, the resulting equation will be of the form

$$h(x_1,x_2,x'_1,x''_1) = 0 \tag{29}$$

which is an uncoupled, nonlinear/linear differential equation with variable coefficients. This can be solved by the existing techniques in the field of nonlinear mechanics to obtain the solution

$$x_1 = p(x_2) \tag{30}$$

Using Eq. (30) along with Eq. (27a) in Eq. (26b) gives the transformed equation in (x_2,t) plane as

$$h_2(x_2,\dot{x}_2,\ddot{x}_2,t) = 0 \tag{31}$$

The solution to this equation, which again is an uncoupled linear/nonlinear differential equation with variable coefficients, is

$$x_2 = F_2(t) \tag{32}$$

Eq. (32) along with Eq. (30) gives the solution

$$x_1 = F_1(t) \tag{33}$$

Equations (32) and (33) represent the response of the given system. Now, the condition on the form of the equations is obtained. For this purpose, assume the governing equations for the coupled nonlinear system with two degrees of freedom, to be of the form

$$\ddot{x}_1 + a_1(x_1,x_2,\dot{x}_1,\dot{x}_2,t) = 0 \tag{34a}$$

$$\ddot{x}_2 + a_2(x_1,x_2,\dot{x}_1,\dot{x}_2,t) = 0 \tag{34b}$$

Choosing x_2 as the new independent variable, and using the differential transformation laws given by Eqs. (27a) and (27b), leads to the equation

$$x''_1 + (a_1 - a_2x'_1)/\dot{x}_2^2 = 0 \tag{35}$$

For this equation, to be uncoupled, the obvious condition is

$$\begin{aligned}(a_1 - a_2x'_1)/\dot{x}_2^2 &= \text{function of } x_1,x'_1 \text{ and } x_2 \text{ only} \\ &= a(x_1,x_2,x'_1)\end{aligned} \tag{36}$$

Hence, the equations of the system, for the method to be applicable, should be of the form,

$$\ddot{x}_1 + a(x_1,x_2,\dot{x}_1/\dot{x}_2)\dot{x}_2^2 + b(x_1,x_2,\dot{x}_1,\dot{x}_2,t) = 0 \tag{37a}$$

$$\ddot{x}_2 + b(x_1,x_2,\dot{x}_1,\dot{x}_2,t)(\dot{x}_2/\dot{x}_1) = 0 \tag{37b}$$

Although the form of the Eqs. (37a) and (37b) indicates that the method is restricted in application to a class of coupled nonlinear equations, the fact that they still represent a wide class of nonlinear problems is brought out by the general functions a and b. Simple pretransformations will further increase the area of applicability of this technique. This is now illustrated by means of a suitable example.

Consider a nonlinear system with two degrees of freedom represented by a pair of simultaneous nonlinear differential equations with variable coefficients in the form,

$$\ddot{y}_1 + (a\dot{y}_1^2/y_1) + p(t)y_2(\dot{y}_1/\dot{y}_2) + q(t)y_1 = 0 \tag{38a}$$

$$\ddot{y}_2 + (b\dot{y}_2^2/y_2) + q(t)y_1(\dot{y}_2/\dot{y}_1) + p(t)y_2 = 0 \tag{38b}$$

The equations in its present form are not amenable to this technique and hence some pretransformations[3] are called for. The following pretransformations are found effective

$$x_1(t) = y_1^{a+1}(t) \tag{39a}$$

$$x_2(t) = y_2^{b+1}(t) \tag{39b}$$

Substituting Eqs. (39a) and (39b) in Eqs. (38a) and (38b), the equations become

$$\ddot{x}_1 + p_1(t)x_2(\dot{x}_1/\dot{x}_2) + q_1(t)x_1 = 0 \tag{40a}$$

$$\ddot{x}_2 + q_1(t)x_1(\dot{x}_2/\dot{x}_1) + p_1(t)x_2 = 0 \tag{40b}$$

where $p_1(t) = p(t)(1 + b)$, $q_1(t) = q(t)(1 + a)$.
Now, choose x_2 (say) as the independent variable. Hence Eq. (40a) becomes

$$x''_1\dot{x}_2^2 + x'_1\ddot{x}_2 + p_1(t)x_2x'_1 + q_1(t)x_1 = 0 \quad (41)$$

Substituting for $\ddot{x}_2$ from Eq. (40b) in Eq. (41), gives the equation

$$x''_1 = 0 \quad (42)$$

which turns out to be a linear differential equation with constant coefficients and hence can be solved to obtain the relation,

$$x_1 = c_1(x_2 + c_2) \quad (43)$$

Use of Eq. (43) in Eq. (40b) gives

$$\ddot{x}_2 + [q_1(t) + p_1(t)]x_2 = -q_1(t)c_2 \quad (44)$$

Eq. (44) is a linear, uncoupled differential equation with variable coefficients and can be solved when $p_1(t)$ and $q_1(t)$, i.e., $p(t)$ and $q(t)$ are specified.

Assuming this to be done, the solution to Eq. (44) becomes

$$x_2(t) = f(t,c_2,c_3,c_4) = y_2^{b+1}(t) \quad (45a)$$

and

$$x_1(t) = c_1[f(t,c_2,c_3,c_4) + c_2] = y_1^{a+1}(t) \quad (45b)$$

These Eqs. together represent the response of the coupled system given by Eqns. (38a) and (38b) and the solutions have the necessary number of arbitrary constants c_1,c_2,c_3 and c_4 which take care of the initial conditions viz., $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$ and $\dot{x}_2(0)$. Thus, the foregoing example clearly illustrates the application of the method.

2.4 Application of the Proposed Method

To illustrate the value of the technique presented in this paper, the practical problem of rigid body motion is considered. This is a third-order coupled system and the equations are similar in form to Eqs. (25a, 25b, and 25c). The Eulers equations for the inertial motion of a rigid body rotating about its center of mass can be written as

$$\dot{\omega}_x + a\omega_y\omega_z = 0 \quad (46a)$$

$$\dot{\omega}_y + b\omega_z\omega_x = 0 \quad (46b)$$

$$\dot{\omega}_z + c\omega_x\omega_y = 0 \quad (46c)$$

where

$$a = \frac{C - B}{A}, \qquad b = \frac{A - C}{B}, \qquad c = \frac{B - A}{C}$$

A, B and C, being the three principal moments of inertia. Let ω_z be the new independent variable.
Therefore

$$\dot{\omega}_x = \omega'_x\dot{\omega}_z \quad (47a)$$

$$\dot{\omega}_y = \omega'_y\dot{\omega}_z \quad (47b)$$

Using Eqs. (47a) and (47b) in Eqs. (46a) to (46c), gives

$$\omega'_x - (a/c)(\omega_z/\omega_x) = 0 \quad (48a)$$

$$\omega'_y - (b/c)(\omega_z/\omega_y) = 0 \quad (48b)$$

Eqs. (48a) and (48b) are uncoupled equations and can be solved by separation of variables to give the relations,

$$\omega_x^2 = (a/c)\omega_z^2 + K_1 \quad (49a)$$

$$\omega_y^2 = (b/c)\omega_z^2 + K_2 \quad (49b)$$

i.e.,

$$\omega_z = [(c/a)(K_1 - \omega_x^2)]^{1/2} = [(c/a)(C_1 - \omega_x^2)]^{1/2}$$

$$\omega_y = [(b/a)(K_1 - \omega_x^2) + K_2]^{1/2} = [(b/a)(C_2 - \omega_x^2)]^{1/2}$$

Hence Eq. (46a) becomes

$$\dot{\omega}_x + [bc(C_1 - \omega_x^2)(C_2 - \omega_x^2)]^{1/2} = 0 \quad (50)$$

Eq. (50) can be solved by separation of variables to obtain the solution in terms of elliptic functions and this checks with the known solutions for this problem which have been obtained by energy considerations[4] through a more tedious process.

$$\int \frac{d\omega_x}{[bc(C_1 - \omega_x^2)(C_2 - \omega_x^2)]^{1/2}} = \int dt + C_3 \quad (51)$$

Hence

$$\omega_x = \alpha Sn(pt + C_3) \quad (52)$$

where α and p are functions of A, B, C and C_1 and C_2. Similar expressions ensue for other angular velocities ω_y and ω_z. Thus the use of the method in a problem of practical interest is brought out in this example.

3. Conclusions

The main feature of this technique is that it helps in uncoupling of the given set of nonlinear differential equations with constant or variable coefficients which may otherwise be a difficult proposition. Although, this uncoupling is not an end in itself, it still forms a progressive step towards the final solution of such nonlinear problems even if the uncoupling were to result in some complex nonlinear equations. As the pretransformations can extend the range of applicability of the method, it is believed that this will find application in a wide class of problems in the field of nonlinear mechanics and other similar fields of nonlinear analysis.

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